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Unit -IV

Quantum Physics

Planck's theory

1. A black body is not only filled up with the radiations but also with a large no. of tiny oscillators. They are atomic dimensions. They are called as atomic oscillators or Planck's oscillators.

2. The frequency of radiation emitted by an oscillator is same as that of oscillator frequency.

3. The oscillator cannot absorb or emit energy in a continuous manner. It can absorb or emit energy in multiples of small units called quantum.

4. The oscillator vibrating with frequency ν can only emit energy in quantum of values $h\nu$. The oscillators vibrating with frequency ν can only have discrete energy values E_n.

$$\varepsilon \propto \nu$$

$$\varepsilon = h\nu$$

h - is Planck's constant ($h = 6.625 \times 10^{-34}$ Js)

$$E_n = nh\nu$$

$$\varepsilon = n\varepsilon$$

n - is positive integer ($n = 1, 2, 3 \ldots$)

Planck's Law of radiation (Derivation)

Average energy $\bar{E}$ per oscillator is given by

$$\bar{E} = \frac{E}{N} \longrightarrow (1)$$

E - Total energy of all oscillators

N - Total no. of oscillators

N_0 - No. of atomic oscillators in ground state

According to Maxwell's energy distribution law, the no. of oscillators with energy ε_n is given by

$$N_n = N_0 e^{-\varepsilon_n/kT} \longrightarrow (2)$$

T - Absolute temperature of the black body

k - Boltzmann's constant

$$N = N_0 + N_1 + N_2 + N_3 \ldots$$

from eqn (2),

$$N = N_0 e^{-\varepsilon_0/kT} + N_0 e^{-\varepsilon_1/kT} + N_0 e^{-\varepsilon_2/kT} + \ldots \longrightarrow (3)$$

$$\varepsilon_n = nh\nu, \quad n = 0, 1, 2, 3 \ldots$$

$$\varepsilon_0 = 0, \quad \varepsilon_1 = h\nu, \quad \varepsilon_2 = 2h\nu$$

Sub values in eqn (3)

$$N = N_0 e^0 + N_0 e^{-h\nu/kT} + N_0 e^{-2h\nu/kT} + \ldots \longrightarrow (4)$$

$$N = N_0 + N_0 e^{-h\nu/kT} + N_0 e^{-2h\nu/kT} + \ldots$$

Put $x = e^{-h\nu/kT}$

$$N = N_0 + N_0 x + N_0 x^2 + N_0 x^3 + \ldots \longrightarrow (5)$$

$$N = N_0 [1 + x + x^2 + \ldots]$$

$$N = N_0\left[\frac{1}{(1-x)}\right] \quad \rightarrow (6)$$

$$N = \frac{N_0}{(1-x)} \quad \rightarrow (7)$$

Total energy of the black body due to all the oscillators is given by

$$E = \varepsilon_0 N_0 + \varepsilon_1 N_1 + \varepsilon_2 N_2 + \cdots \quad \rightarrow (8)$$

Subst. $\varepsilon_0, \varepsilon_1, \varepsilon_2$. & $N_0, N_1, N_2 \ldots$ in eqn (7)

$$E = 0 \times N_0 + h\nu N_0 e^{-h\nu/kT} + 2h\nu N_0 e^{-2h\nu/kT} + \cdots$$

$$E = h\nu N_0 e^{-h\nu/kT} + 2h\nu N_0 e^{-2h\nu/kT} + \cdots$$

put $x = e^{-h\nu/kT}$

$$E = h\nu N_0 x + 2h\nu N_0 x^2 + \cdots \quad \rightarrow (9)$$

$$E = h\nu N_0 x\left[1 + 2x + \cdots\right]$$

$$E = \frac{h\nu N_0 x}{(1-x)^2} \quad \rightarrow (10)$$

sub eqns (7) & (10) in (1), we get

$$\bar{E} = \frac{h\nu N_0 x/(1-x)^2}{N_0/(1-x)}$$

$$= \frac{h\nu N_0 x}{(1-x)^2} \times \frac{(1-x)}{N_0}$$

$$= \frac{h\nu x}{(1-x)} = \frac{h\nu x}{x\left(\frac{1}{x}-1\right)}$$

$$= \frac{h\nu}{\left(\frac{1}{x}-1\right)}$$

$$x = e^{-h\nu/kT}$$

$$\bar{E} = \frac{h\nu}{e^{+h\nu/kT} - 1} \longrightarrow (11)$$

No. of oscillators per unit volume in the wavelength rang λ & $\lambda + d\lambda$

$$\frac{8\pi d\lambda}{\lambda^4} \longrightarrow (12)$$

The energy density of radiation in the wavelengths λ and $\lambda + d\lambda$ is given by

$$E_\lambda \, d\lambda = \frac{8\pi d\lambda}{\lambda^4} \times \frac{h\nu}{e^{h\nu/kT} - 1}$$

$$= \frac{8\pi d\lambda}{\lambda^4} \times \frac{hc/\lambda}{e^{h\nu/kT} - 1} \qquad [c = \nu\lambda \quad \nu = c/\lambda]$$

$$E_\lambda = \frac{8\pi hc}{\lambda^5 (e^{h\nu/kT} - 1)} \longrightarrow (13)$$

Wien's displacement law

λ is very small, ν is large so as $\frac{h\nu}{kT} >> 1$

hence $e^{h\nu/kT} - 1 \approx e^{h\nu/kT}$

W.K.T

$$E_\lambda = \frac{8\pi hc}{\lambda^5 (e^{h\nu/kT} - 1)}$$

Sub in above eqn, we get.

$$E_\lambda = \frac{8\pi hc}{\lambda^5 \, e^{h\nu/kT}} \longrightarrow (14)$$

Rayleigh Jean's law from planck's law

λ is very large , ν is small

$$e^{h\nu/kT} = 1 + \frac{h\nu}{kT}$$

sub in eqn (13) , we get

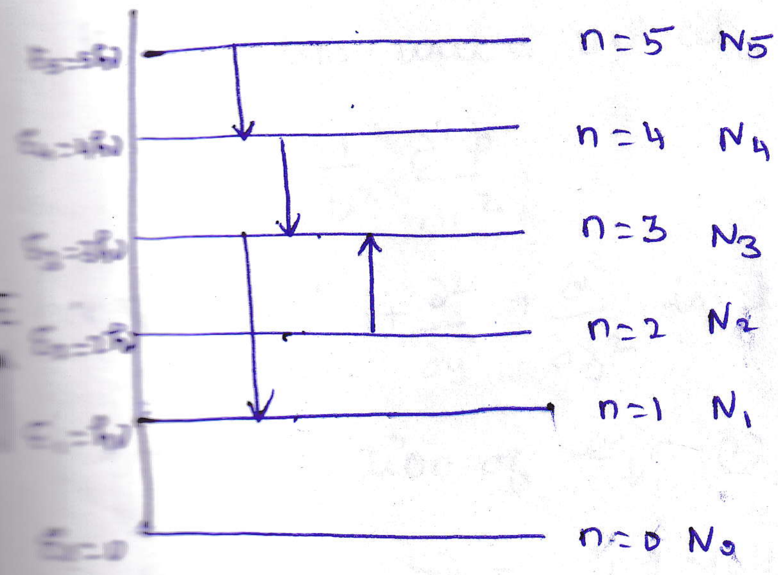
$$E_\lambda = \frac{8\pi hc}{\lambda^5 \, e^{h\nu/kT} - 1}$$

$c = \nu\lambda$

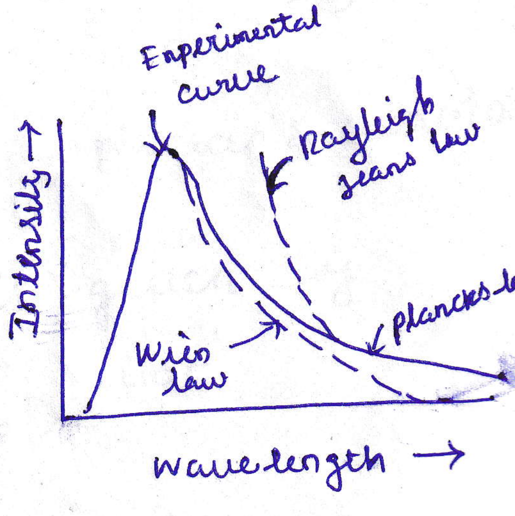
$\nu = \frac{c}{\lambda}$

$$= \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{1 + \frac{h\nu}{kT} - 1}$$

$$E_\lambda = \frac{8\pi kT}{\lambda^4} \longrightarrow (15)$$



Energy diagram for planck's oscillator frequency ν



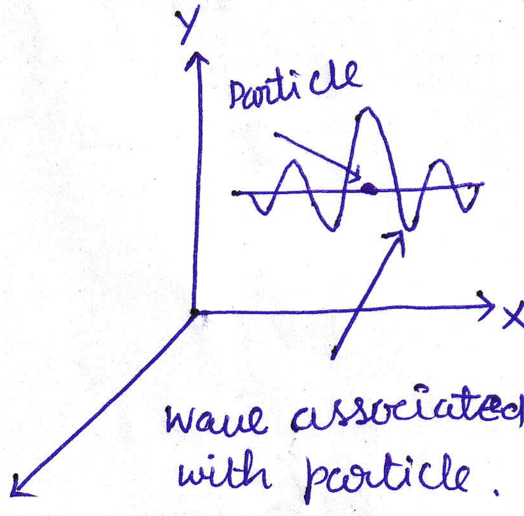
Planck's formula is found to agree remarkably well with experimental observations for smaller and longer wavelengths. So, it is an established formula for the validity of quantum hypothesis

Schroedinger Time independent wave equation (Derivation)

consider a wave associated with a moving particle.

Let x, y, z be the coordinates of the particle

ψ is wave function for de-Broglie's waves at any given instant of time t.



The classical differential equation for wave motion is given by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \longrightarrow ①$$

where v is wave volicity

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \longrightarrow ②$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian's operator

The solution of eqn ② is given by

$$\psi(x, y, z, t) = \psi_0(x, y, z)e^{-i\omega t}$$

$$\psi = \psi_0 e^{-i\omega t} \longrightarrow ③$$

eqn ③ Differentiating w.r.to t

$$\frac{\partial\psi}{\partial t} = -i\omega\psi_0 e^{-i\omega t}$$

Again differentiating w.r.to t

$$\frac{\partial^2\psi}{\partial t^2} = (-i\omega)(-i\omega)\psi_0 e^{-i\omega t}$$

$$\frac{\partial^2\psi}{\partial t^2} = i^2\omega^2\psi_0 e^{-i\omega t}$$

$i^2 = -1$

$\psi = \psi_0 e^{-i\omega t}$

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi \longrightarrow (4)$$

sub eqn (4) in eqn (2)

$$\nabla^2\psi = -\frac{\omega^2}{v^2}\psi$$

$$\nabla^2\psi + \frac{\omega^2}{v^2}\psi = 0 \longrightarrow (5)$$

W.K.T

$$\omega = 2\pi\nu$$

$$\nu = \frac{v}{\lambda}$$

$\frac{\omega}{\nu} = \frac{2\pi}{\lambda}$

$$\frac{\omega}{v} = \frac{2\pi}{\lambda} \longrightarrow (6)$$

$$\frac{\omega^2}{v^2} = \frac{4\pi^2}{\lambda^2} \longrightarrow (7)$$

sub. eqn (7) in eqn (5)

$$\nabla^2\psi + \frac{4\pi^2}{\lambda^2}\psi = 0 \longrightarrow (8)$$

W.K.T $\lambda = \frac{h}{mv}$

$$\lambda^2 = \frac{h^2}{m^2v^2} \longrightarrow (9)$$

Sub eqn (9) in (8)

$$\nabla^2\psi + \frac{4\pi^2}{\frac{h^2}{m^2v^2}}\psi = 0$$

$$\nabla^2\psi + \frac{4\pi^2 m^2 v^2}{h^2}\psi = 0 \longrightarrow (10)$$

Total energy = Potential energy + kinetic energy

$$E = V + \frac{1}{2}mv^2$$

$$E - V = \frac{1}{2}mv^2$$

$$2(E-V) = mv^2$$

$$2m(E-V) = m^2v^2 \longrightarrow (11)$$

Sub eqn (11) in eqn (10)

$$\nabla^2\psi + \frac{4\pi^2}{h^2}2m(E-V)\psi = 0$$

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}(E-V)\psi = 0 \longrightarrow (12)$$

W.K.T

$$\hbar = \frac{h}{2\pi}$$

$$\hbar^2 = \frac{h^2}{4\pi^2} \Rightarrow h^2 = \hbar^2 4\pi^2 \longrightarrow (13)$$

Sub eqn (13) in eqn (12)

$$\nabla^2\psi + \frac{8\pi^2 m}{\hbar^2 4\pi^2}(E-V)\psi = 0$$

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0 \longrightarrow (14)$$

eqn (14) is mult by $-\hbar^2/2m$, we get

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \longrightarrow (15)$$

The above eqn is Schroedinger time

Schroedinger time dependent wave equation

Schroedinger time dependent wave eqn. is derived from schroedinger time independent wave equation.

The solu. is

$$\psi(x,y,z,t) = \psi_0(x,y,z)\, e^{-i\omega t} \longrightarrow ①$$

Diff eqn ① w.r.to t $\qquad [\because \psi = \psi_0 e^{-i\omega t}]$

$$\frac{\partial \psi}{\partial t} = -i\omega \psi_0 e^{-i\omega t} \longrightarrow ②$$

$$\frac{\partial \psi}{\partial t} = -i(2\pi\nu)\psi_0 e^{-i\omega t}$$

$$\frac{\partial \psi}{\partial t} = -2\pi i \nu \psi \longrightarrow ③$$

$$\frac{\partial \psi}{\partial t} = -2\pi i \frac{E}{h}\psi \qquad \because E = h\nu, \quad \nu = \frac{E}{h}$$

$$\frac{\partial \psi}{\partial t} = -i \frac{E\psi 2\pi}{h} \qquad \frac{2\pi}{h} = \frac{1}{\hbar}$$

$$\frac{\partial \psi}{\partial t} = -\frac{iE}{\hbar}\psi \longrightarrow ④$$

mult. i on both sides of eqn ④

$$i\frac{\partial \psi}{\partial t} = -i^2 \frac{E}{\hbar}\psi \qquad [i^2 = -1, \; -i^2 = +1]$$

$$i\frac{\partial \psi}{\partial t} = \frac{E}{\hbar}\psi$$

$$i\hbar \frac{\partial \psi}{\partial t} = E\psi \longrightarrow (5)$$

Schroedinger time independent wave eqn. is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

Sub for $E\psi$ from eqn (5)

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial \psi}{\partial t} \longrightarrow (6)$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial \psi}{\partial t} \longrightarrow (7)$$

$$H\psi = E\psi \longrightarrow (8)$$

where

$H = \left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)$ is Hamiltonian operator

$E = i\hbar\frac{\partial}{\partial t}$ is energy operator

The eqn (7) is known as schroedinger time dependent wave equation.

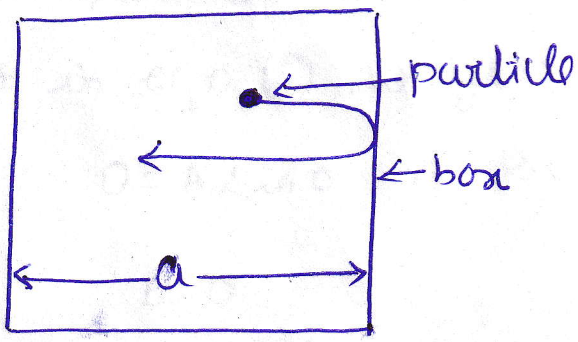
In eqn (7), $\frac{\partial \psi}{\partial t}$ representing time.

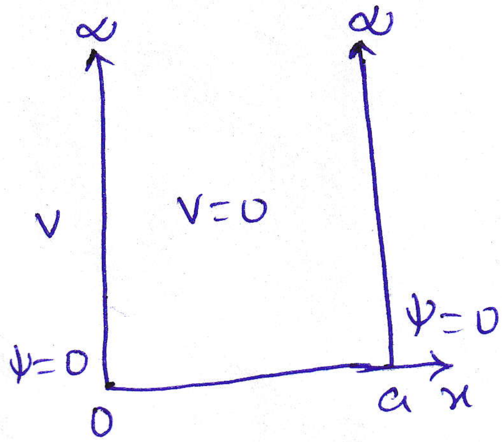
Applications of schrodinger equation

particle in a one-dimensional Rigid box

consider a particle of mass 'm' moving between two rigid walls of a box at $x=0$ and $x=a$ along x axis.

The particle is travelling front & back and the potential energy (V) of particle inside the box is constant. (ie) zero.





The potential function is given by

$V(x) = 0$ for $0 < x < a$

$V(x) = \infty$ for $0 \geq x \geq a$

The value of ψ is calculated within the box

Schrodinger's wave eqn in one-dimensional box

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0 \longrightarrow (1)$$

Since $V=0$,

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \longrightarrow (2)$$

$$\frac{2mE}{\hbar^2} = k^2 \text{, sub in eqn (2)}$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \longrightarrow (3)$$

The general soln. of eqn (3)

$$\psi(x) = A \sin kx + B \cos kx \longrightarrow (4)$$

A & B are two unknown constant

Boundary condition (i)

$$\psi = 0 \text{ at } x = 0$$

Sub in eqn (4), we get

$$0 = A \sin 0 + B \cos 0$$

$$B = 0$$

Boundary condition (ii)

$$\psi = 0 \text{ at } x = a$$

$$0 = A \sin ka + 0$$

$$A \sin ka = 0$$

$$A \neq 0 \text{, so } \sin ka = 0$$

$$ka = n\pi$$

$$k = \frac{n\pi}{a} \longrightarrow (5)$$

Sub in eqn

$$k^2 = \frac{n^2\pi^2}{a^2}$$

$$k^2 = \frac{n^2\pi^2}{\ldots} = \frac{2mE}{\ldots}$$

$$k^2 = \frac{2m\, 4\pi^2\, E}{h^2}$$

$$k^2 = \frac{8\pi^2 mE}{h^2} \longrightarrow (7)$$

equate eqn (6) & (7)

$$\frac{n^2\pi^2}{a^2} = \frac{8\pi^2 mE}{h^2}$$

Energy of the particle

$$E_n = \frac{n^2 h^2}{8ma^2} \longrightarrow (8)$$

$$\psi_n(x) = A \sin \frac{n\pi x}{a} \longrightarrow (9)$$

$n = 1, 2, 3 \ldots$

* The particle in a box cannot possess any arbitary amount of energy. It can only have discrete energy values.

* Each value of E_n is called eigen value and corresponding ψ_n is called eigen function.

Normalisation of wave function.

Probability density is given by $\psi^* \psi$

$$\psi_n(x) = A \sin \frac{n\pi x}{a}$$

$$\psi^* \psi = A \sin \frac{n\pi x}{a} \times A \sin \frac{n\pi x}{a}$$

$$\psi^* \psi = A^2 \sin^2\left(\frac{n\pi x}{a}\right) \longrightarrow (10)$$

$$\int_0^a \psi^* \psi \, dx = 1 \longrightarrow (11)$$

Sub eqn (10) in eqn (11)

$$\int_0^a A^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = 1$$

$$A^2 \int_0^a \frac{\left[1 - \cos\left(\frac{2n\pi x}{a}\right)\right]}{2} dx = 1$$

$$\frac{A^2}{2}\left[\int_0^a dx - \int_0^a \cos\left(\frac{2n\pi x}{a}\right) dx\right] = 1$$

$$\frac{A^2}{2}\left[(x)_0^a - 0\right] = 1 \qquad \left[\int_0^a \cos\left(\frac{2n\pi x}{a}\right) dx = 0\right]$$

$$\frac{A^2}{2}(x)_0^a = 1$$

$$\frac{A^2 a}{2} = 1 \quad (\text{or}) \quad A^2 = \frac{2}{a}$$

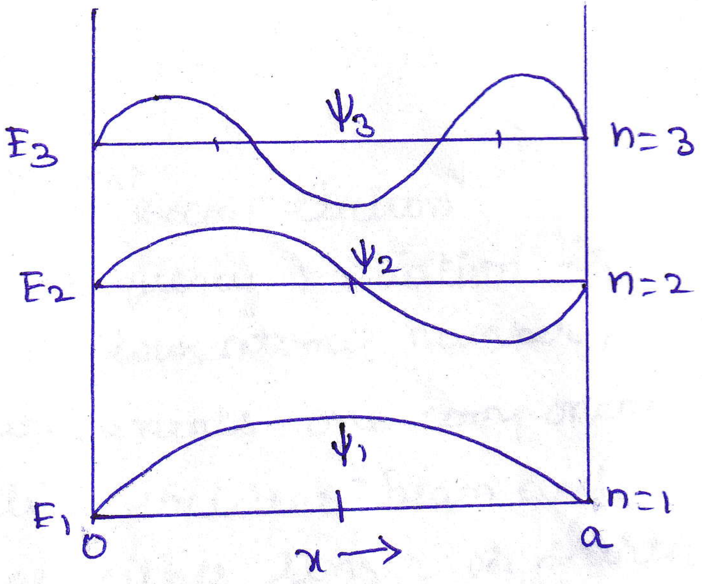
$$A = \sqrt{\frac{2}{a}} \longrightarrow (12)$$

$$\psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \longrightarrow (13)$$

eqn (13) is known as normalised ~~eqn~~ eigen function.

Special cases

Case (i) $n=1$



$$E_n = \frac{n^2 h^2}{8ma^2}$$

$$E_1 = \frac{h^2}{8ma^2}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

$\psi_1(x)$ is maximum exactly at middle of the box.

Case (ii) $n=2$

$$E_2 = \frac{4h^2}{8ma^2} = 4E_1$$

$$\psi_2(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$$

$\psi_2(x)$ is maximum at quarter distance from either sides of the box

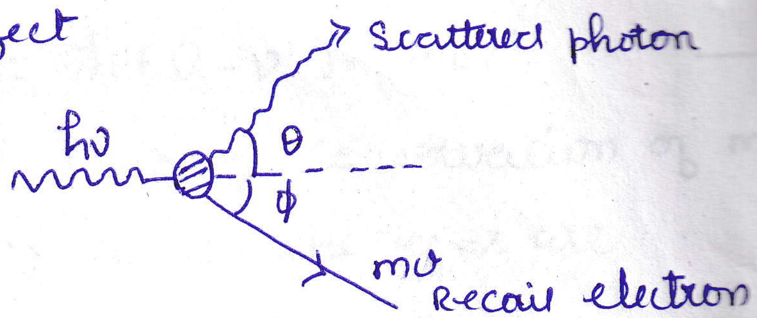
Case (iii) $n=3$

$$E_3 = \frac{9h^2}{8ma^2} = 9E_1$$

$$\psi_3(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right)$$

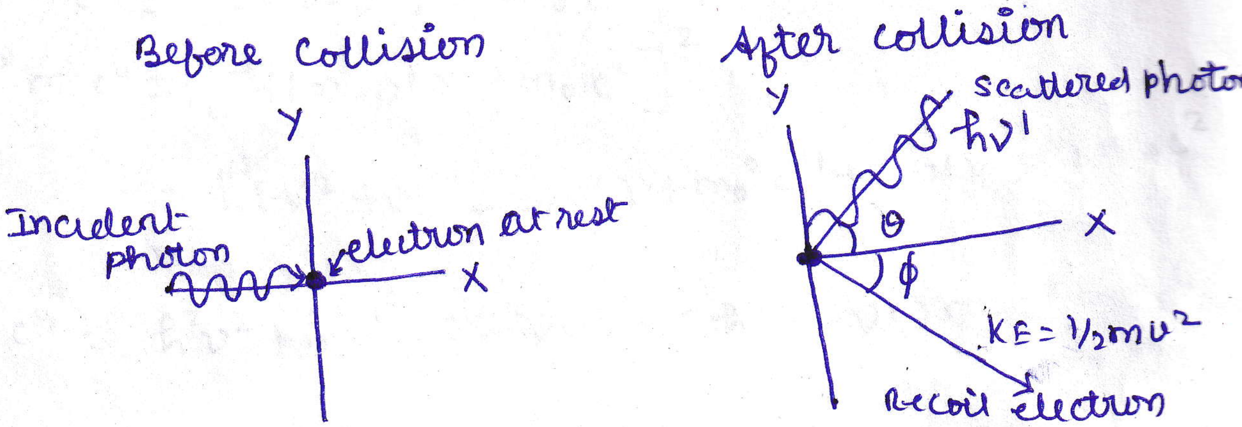
$\psi_3(x)$ is maximum exactly ~~at~~ middle and one-sixth distance from either sides of the box.

Compton effect



When a beam of high frequency radiation is scattered by a substance of low atomic number which results in two components. One component has same wavelength like incident beam and the other component has either longer or shorter. This effect is called compton effect and the shift is called compton shift.

Explanation



When a photon of energy hν collides with a free electron of the scattering substance, it transfers energy to the electron. During the collision it undergoes to the laws of conservation of energy and momentum.

Applying the law of conservation of energy we get the eqn

$$h\nu + m_0c^2 = h\nu' + mc^2$$

Energy before collision = Energy after collision

$$mc^2 = h(\nu - \nu') + m_0c^2 \longrightarrow (1)$$

Applying the law of conservation of momentum along x & y axes the eqns are

Momentum before collision = Momentum after collision

$$muc\cos\phi = h(\nu - \nu'\cos\theta) \longrightarrow (2)$$

$$muc\sin\phi = h\nu'\sin\theta \longrightarrow (3)$$

Squaring and adding the eqn (2) & (3) we get

$$m^2u^2c^2 = h^2(\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta) \longrightarrow (4)$$

Squaring the eqn (1)

$$mc^2 = h(\nu-\nu') + m_0c^2$$

$$m^2c^4 = [h(\nu-\nu') + m_0c^2]^2$$

$$= h^2(\nu^2 + \nu'^2 - 2\nu\nu') + m_0^2c^4 + 2h(\nu-\nu')m_0c^2$$

$$m^2c^4 = h^2\nu^2 + h^2\nu'^2 - 2h^2\nu\nu' + 2h(\nu-\nu')m_0c^2 + m_0^2c^4 \rightarrow (5)$$

Sub eqn (4) from eqn (5) we get

$$m^2c^2(c^2 - u^2) = -2h^2\nu\nu'(1-\cos\theta) + 2h(\nu-\nu')m_0c^2 + m_0^2c^4 \rightarrow (6)$$

According to theory of relativity

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \rightarrow (7)$$

Squaring the eqn (7) we get

$$m^2c^2(c^2 - u^2) = m_0^2c^4 \rightarrow (8)$$

equating eqn ⑥ & ⑧ we get

$$m_0^2 c^4 = -2h^2\nu\nu'(1-\cos\theta) + 2h(\nu-\nu')m_0c^2 + m_0^2c^4$$

$$2h(\nu-\nu')m_0c^2 = 2h^2\nu\nu'(1-\cos\theta)$$

$$(\nu-\nu')m_0c^2 = h\nu\nu'(1-\cos\theta)$$

$$\frac{\nu-\nu'}{\nu\nu'} = \frac{h}{m_0c^2}(1-\cos\theta)$$

$$\frac{\nu}{\nu\nu'} - \frac{\nu'}{\nu\nu'} = \frac{h}{m_0c^2}(1-\cos\theta)$$

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0c^2}(1-\cos\theta) \rightarrow (9)$$

W.K.T

$$c = \nu\lambda$$

$$\nu = \frac{c}{\lambda}, \quad \frac{1}{\nu} = \frac{\lambda}{c} \rightarrow (9a)$$

Sub eqn ⑩ in eqn ⑨ we get

$$\frac{\lambda'}{c} - \frac{\lambda}{c} = \frac{h}{m_0c^2}(1-\cos\theta)$$

$$\frac{1}{c}(\lambda'-\lambda) = \frac{h}{m_0c^2}(1-\cos\theta)$$

$$\lambda'-\lambda = \frac{h}{m_0c}(1-\cos\theta)$$

$$\lambda'-\lambda = d\lambda = \Delta\lambda$$

$$d\lambda = \frac{h}{m_0c}(1-\cos\theta) \rightarrow (10)$$

Case - 1

If $\theta = 0^\circ$, $\cos\theta = 1$

$$d\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

$$d\lambda = \frac{h}{m_0 c}(1-1).$$

$$\boxed{d\lambda = 0}$$

Case : 2

If $\theta = 90^\circ$, $\cos 90^\circ = 0$

$$d\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

$$= \frac{h}{m_0 c}(1-0)$$

$$d\lambda = \frac{h}{m_0 c}$$

$$\boxed{d\lambda = 0.0243\,\mathring{A}}$$

Case : 3

If $\theta = 180^\circ$, $\cos 180^\circ = -1$

$$d\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

$$= \frac{h}{m_0 c}(1+1)$$

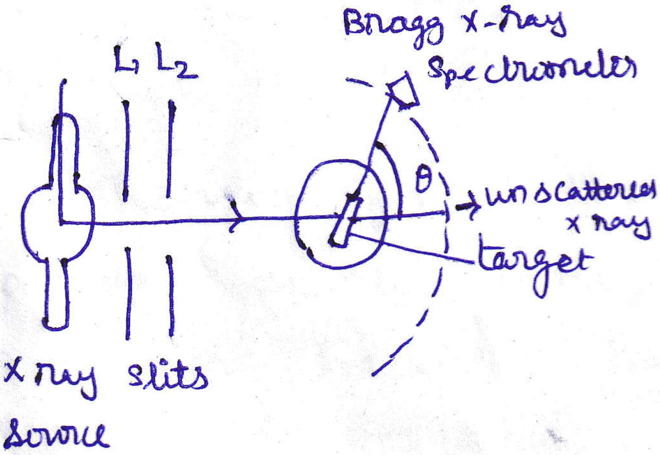
$$d\lambda = \frac{2h}{m_0 c}$$

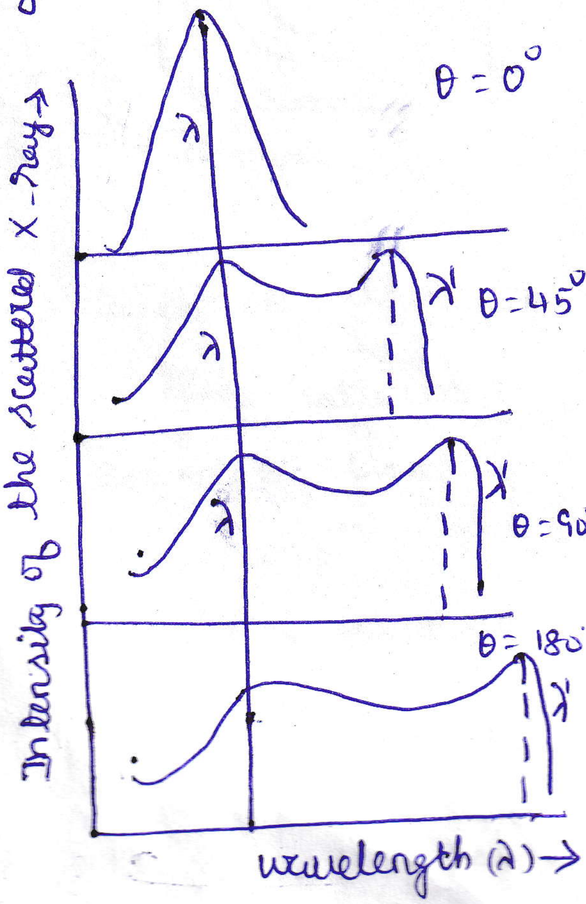
$$\boxed{d\lambda = 0.0486\,\mathring{A}}$$

The difference in wave length is called compton wavelength.

Experimental verification of compton effect.

A beam of monochromatic x-ray is made to incident on a target material for various scattering angles intensity of scattered x-ray is measured.





h - Planck's constant

c - speed of light

λ - unmodified line

λ' - modified line.

The difference between two peaks on the wavelength axis gives the compton shift.

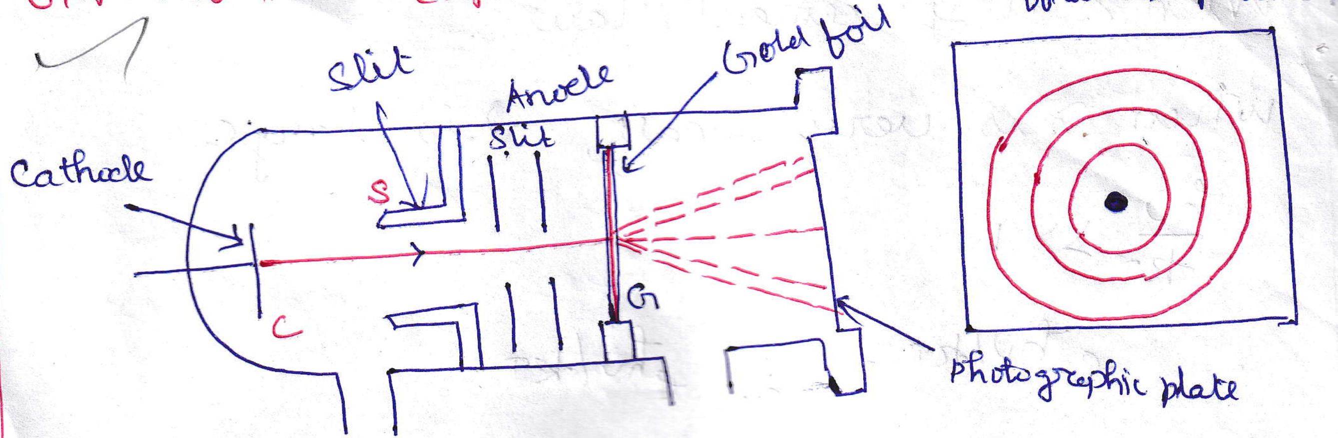
$$d\lambda = \frac{h}{m_0 c}(1 - \cos\theta)$$

The change in wavelength $d\lambda = 0.0243$ A° at 90° is compared with theoritical value.

Thus, compton effect is experimentally verified.

The value $d\lambda = 0.0243$ Å is known as Compton wave length.

G.P. Thomson Experiment



Thomson found that the electron diffraction patterns exactly analogous to X-ray patterns. Moreover, he was able to determine the wavelength associated with electrons.

1. It consist of discharge tube with electrons
2. The electrons are accelerated by a potential 50,000 volts.
3. The accelerated electrons are passed through slit and then incident on gold foil G.
4. The emerged electrons are recorded by a photographic plate.
5. After developing the photographic plate, a symmetrical pattern consisting of concentric ring about a central spot diffraction pattern obtained.
6. The pattern can only be produced by waves and not by the particles. Hence, electrons behaved like waves.
7. The wave length of electrons depends only on the accelerating voltage not depends on nature target material.